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Strictly positive definite functions in $\mathbb{R}^d$

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Abstract

We give a sufficient condition for strictly positive definiteness in $\mathbb{R}^d$. The result is based on the question how sparse subsets of $\mathbb{R}^d$ can be to guarantee linear independence of the exponentials.

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Interpolation by positive definite functions has become a widely used technique in approximation theory and spatial statistics. The basic model is defined as linear combination of translates of a given positive definite function, called basis function. Setting up the collocation matrix for the problem, one has to assume the matrix to be invertible. This is guaranteed if the basis function is assumed to be strictly positive definite. Hereby, a function $\varphi : \mathbb{R}^d \to \mathbb{C}$ is called strictly positive definite if for arbitrary distinct points $x_1, \ldots, x_m \in \mathbb{R}^d$ and complex coefficients $c_1, \ldots, c_m$ the inequality

$$\sum_{k,l=1}^{m} c_k \overline{c_l} \varphi(x_k - x_l) > 0 \tag{1}$$

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holds true. Using Bochner's characterization of continuous positive definite functions as Fourier transforms of bounded non-negative measures it is straightforward to see that verifying condition (1) reduces to checking whether the exponentials are linearly independent over a certain subset of $\mathbb{R}^d$.

Characterizations of strictly positive definite functions in various settings are given by Chen et al. [1], Pinkus [2,3], and Sun [4]. The latter deals with radial functions in $\mathbb{R}^d$. The question of linear independence of the exponentials is not treated in detail there. This gives us the motivation to ask how small a set in $\mathbb{R}^d$ can be to guarantee linear independence. In the present communication we show that such sets can be indeed very small. To state the result we need the following inductive definition.

Definition. A subset A of S^1 will be called admissible if it contains infinitely many points. A subset A of S^{d-1}, $d \geqslant 3$, will be called admissible if there are infinitely many distinct unit vectors $\{v_n\}_{n=1}^{\infty}$ such that for each k the set $A \cap v_k^{\perp}$ in $S^{d-1} \cap v_k^{\perp}$ (which can be identified with S^{d-2}) is admissible.

Remark. For $d = 2$ the definition is explicit. For higher dimensions admissible subsets are roughly those which are big enough that after intersections with infinitely many spheres of dimension $d-1$ the set is still admissible.

Observe that the whole sphere is obviously admissible. Although, an admissible set in S^{d-1} can be much smaller, for instance a countable set with just one accumulation point.

For a given vector v let P_v denote the projection onto the orthogonal complement of v.

Lemma. *Given an admissible subset* $A \subset S^{d-1}$. *Let* $x_1, x_2, \ldots, x_m$ *be distinct vectors in* $\mathbb{R}^d$. *There is a vector* $v \in \mathbb{R}^d$ *such that the vectors* $P_v(x_1), P_v(x_2), \ldots, P_v(x_m)$ *are distinct and the subset* $A \cap v^{\perp} \subset S^{d-1} \cap v^{\perp}$ *is admissible.*

Proof. By assumption there exist vectors v_k as in the definition of admissibility. Consider the set of all differences $\{x_i - x_j\}_{i \neq j}$. Since they determine only finitely many directions there is a number k such that v_k is not parallel to any $x_i - x_j$. Hence $P_{v_k}(x_i - x_j) \neq 0$. □

Corollary. *Let* $A \subset S^{d-1}$ *be admissible and* $x_1, x_2, \ldots, x_m$ *be distinct vectors in* $\mathbb{R}^d$. *Then there are orthogonal unit vectors* v *and* w *such that the numbers* $\langle x_1, v\rangle, \langle x_2, v\rangle, \ldots, \langle x_m, v\rangle$ *are distinct and the set* $A \cap \operatorname{span}\{v, w\}$ *is infinite.*

Proof. For $d = 2$ the statement follows directly from the lemma. For $d \geqslant 3$ we use the lemma in order to decrease the dimension by one. The proof can then be completed by induction. □

The main result of the paper is the following.

Theorem. *Assume* μ *is a probability measure on* $\mathbb{R}^d$ *such that there are* $r > 0$ *and* $v_0 \in \mathbb{R}^d$ *such that the set* $(v_0 + r^{-1} \operatorname{supp} \mu) \cap S^{d-1}$ *is admissible. Then the Fourier transform of* μ, *i.e.*,

$$\varphi(x) = \int_{\mathbb{R}^d} e^{i\langle y,x\rangle}\, d\mu(y), \quad x \in \mathbb{R}^d,$$

is a strictly positive definite function.

Remark. Observe that if the support of μ contains a sphere rS^{d-1} for some $r > 0$ then the assumptions are satisfied. In particular, if the measure μ is rotation invariant and not concentrated at the origin the function φ is strictly positive definite.

Proof. An affine transformation of the measure μ does not affect the statement, so we may assume that $r = 1$ and $v_0 = 0$. Let $x_1, x_2, \ldots, x_m$ be distinct points in $\mathbb{R}^d$ and let $c_1, c_2, \ldots, c_m$ be complex numbers. Assume that

$$\sum_{k,l=1}^{m} c_k \overline{c_l} \varphi(x_k - x_l) = 0.$$

We have to show that $c_1 = c_2 = \cdots = c_m = 0$. The proof will go by induction on m. Making use of Fourier transform, we get

$$\int_{\mathbb{R}^d} \left| \sum_{k=1}^{m} c_k e^{i\langle y, x_k \rangle} \right|^2 d\mu(y) = 0.$$

Therefore,

$$\sum_{k=1}^{m} c_k e^{i\langle y, x_k \rangle} = 0, \quad y \in \operatorname{supp} \mu. \tag{2}$$

Let v and w be vectors satisfying the statement of the corollary applied to the set $A = \operatorname{supp} \mu \cap S^{d-1}$. Since $A \cap \operatorname{span}\{v, w\}$ is infinite we have $v \sin t + w \cos t \in A$ for infinitely many $t \in [0, 2\pi)$. Let $a_k = \langle x_k, v \rangle$ and $b_k = \langle x_k, w \rangle$. By the corollary the numbers a_k are all distinct. We may assume that $a_1 < a_2 < \cdots < a_m$. By (2) we obtain

$$\sum_{k=1}^{m} c_k e^{i a_k \sin t + i b_k \cos t} = 0$$

for infinitely many $t \in [0, 2\pi)$. Since the function on the left hand side depends analytically on t the equality is valid for any complex number t. In particular, let t be purely imaginary, i.e., $t = -iu$. Then $\cos t = \cosh u$ and $\sin t = -i \sinh u$. Hence

$$\sum_{k=1}^{m} c_k e^{a_k \sinh u + i b_k \cosh u} = 0, \quad u \in \mathbb{R}.$$

Divide both sides of the equation by $e^{a_m \sinh u + i b_m \cosh u}$ to obtain

$$\sum_{k=1}^{m} c_k e^{(a_k - a_m) \sinh u + i(b_k - b_m) \cosh u} = 0.$$

Now taking the limit $u \to +\infty$ and using the fact that $a_k < a_m$ for $k < m$ gives $c_m = 0$. □

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